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Discontinuous Automorphisms of the Proper Galilei and Euclidean Groups

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It is shown that discontinuous automorphisms of the proper Galilei and Euclidean groups (i.e., without space inversions) exist in three dimensions, but not in higher dimensions.

1. INTRODUCTION

There is a vast literature in mathematical physics and algebra on automorphisms of classical groups. See, for instance, the expository articles by Levy-Leblond (1971) and O'Meara (1971). Levy-Leblond (1971) posed the problem of finding the abstract automorphisms of the *proper* Galilei group; that is, of the Galilei group without space inversions. Adeleke (1980) also sought the solution of the problem for the proper Euclidean group; in the same paper, Adeleke used the partial solutions obtained to unify several theories of symmetry in mechanics. The theory of symmetry in mechanics plays an important role in determining the response of materials to stretches; that is, in determining their constitutive equations.

Besides this, Marmo and Whiston (1972) indicated other uses of the determination of the automorphisms of the Galilei and Euclidean groups. They also showed that if inversions are included, all the automorphisms of the Galilei group are continuous. Before this work, Michel (1967) had shown the same result in the relativistic case. Furthermore, the paper of Adeleke (1982) showed that discontinuous automorphisms of the proper Galilei group exist if and only if they exist for the Euclidean group in three dimensions. It becomes clear that Levy-Leblond (1971) excluded space inversions in his definition of the proper Galilei group if one considers his Section II.D.

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In this paper, I show that there are numerous discontinuous automorphisms of the three-dimensional Euclidean group and thus of the Galilei group, as numerous as the subsets of the real numbers, $2^{2^{\aleph_0}}$, to be exact. These automorphisms arise from the derivations of the field of real numbers. Tits (1970, Section 11.3) gave an example of a discontinuous automorphism for the semidirect product of the general linear group $GL_n(K)$ in n dimensions with the additive group of $n \times n$ matrices. I observe below that if one uses the quaternion representation of the rotations and the *form* of Tits' example, one obtains discontinuous automorphisms of the Euclidean group in three dimensions. This is the main contribution in the part of this work which deals with the construction of discontinuous automorphisms. Besides this, however, I show that all automorphisms of the Euclidean group in higher dimensions are continuous. The main results are stated precisely in Section 2. Section 3 contains the notations and definitions; Section 4 contains the proofs of the results in dimension 3, while Section 5 deals with higher dimensions.

2. RESULTS

See Section 3 for details of notations.

Theorem 1. Let $\mathscr{E}_3^+$ be the Euclidean group in three dimensions with no space inversions, and let $\mathscr{G}$ be the corresponding Galilei group. Then $\mathscr{E}_3^+$ and $\mathscr{G}$ both have discontinuous automorphisms.

Theorem 2. For $n \geq 4$, all automorphisms of $\mathscr{E}_n^+$ are continuous.

Theorem 3. The cardinality of the set of all discontinuous automorphisms of $\mathscr{E}_3^+$ is $2^{2^{\aleph_0}}$, where $\aleph_0$ is the cardinality of the set of all integers. (Note that $2^{\aleph_0}$ is the cardinality of the set of all real numbers, and $2^{2^{\aleph_0}}$ is the cardinality of the collection of all subsets of the set of real numbers.)

Lemma 1. All discontinuous automorphisms of $\mathscr{E}_3^+$ arise from derivations of $\mathbb{R}$.

Every such automorphism is a composite function $B \cdot A$, where:

(2.1)(i). We have

$$([\boldsymbol{\xi}], \mathbf{a}) \overset{A}{\mapsto} ([\boldsymbol{\xi}], \boldsymbol{\xi}\, d(\bar{\boldsymbol{\xi}}) + \mathbf{a})$$

$$([\boldsymbol{\xi}], \mathbf{a}) \overset{B}{\mapsto} ([\boldsymbol{\eta}_0 \boldsymbol{\xi} \bar{\boldsymbol{\eta}}_0], k_0 \boldsymbol{\eta}_0 \mathbf{a} \bar{\boldsymbol{\eta}}_0 + \boldsymbol{\eta}_0 \mathbf{b}_0 \bar{\boldsymbol{\eta}}_0 - \mathbf{b}_0)$$

(2.1)(ii). $[\boldsymbol{\xi}]$ is a (quaternion representation of a) rotation; $\mathbf{a} \in \mathbb{R}^3$; $\mathbf{b}_0$ is a fixed constant in $\mathbb{R}^3$; $[\boldsymbol{\eta}_0]$ is a fixed rotation; and k_0 is a fixed, nonzero constant in $\mathbb{R}$.

(2.1)(iii). d is a derivation of $\mathbb{R}$.

(2.1)(iv). Composition is read from right to left in $B \cdot A$ and below.

Lemma 2. All discontinuous automorphisms of $\mathscr{G}$ also arise from derivations of $\mathbb{R}$. Every such automorphism is a composite function $C \cdot D$, where, in addition to (2.1)(ii) and (2.1)(iii) above, we have the following:

(2.1)(v). We have

$$[(\boldsymbol{\xi}], \mathbf{v}, \mathbf{a}, \mathrm{t}) \overset{C}{\mapsto} ([\boldsymbol{\xi}], \mathbf{v}, \boldsymbol{\xi}\, d(\bar{\boldsymbol{\xi}}) + \mathbf{a}, \mathrm{t})$$

$$\begin{aligned}([\boldsymbol{\xi}], \mathbf{v}, \mathbf{a}, \mathrm{t}) \overset{D}{\mapsto} ([\boldsymbol{\eta}_0\boldsymbol{\xi}\bar{\boldsymbol{\eta}}_0], & \ k_0\boldsymbol{\eta}_0\mathbf{v}\bar{\boldsymbol{\eta}}_0 + \boldsymbol{\eta}_0\boldsymbol{\xi}\mathbf{b}_0\bar{\boldsymbol{\xi}}\bar{\boldsymbol{\eta}}_0 - \boldsymbol{\eta}_0\mathbf{b}_0\bar{\boldsymbol{\eta}}_0, \\ & p_0k_0\boldsymbol{\eta}_0\mathbf{a}\bar{\boldsymbol{\eta}}_0 + q_0\boldsymbol{\eta}_0\mathbf{v}\bar{\boldsymbol{\eta}}_0 - k_0t\boldsymbol{\eta}_0\mathbf{m}_0\bar{\boldsymbol{\eta}}_0 \\ & + \boldsymbol{\eta}_0\boldsymbol{\xi}\mathbf{r}_0\bar{\boldsymbol{\xi}}\bar{\boldsymbol{\eta}}_0 - \boldsymbol{\eta}_0\mathbf{r}_0\bar{\boldsymbol{\eta}}_0, \ k_0t)\end{aligned}$$

where p_0 is a fixed nonzero constant in $\mathbb{R}$, q_0 is a fixed constant in $\mathbb{R}$, and $\mathbf{m}_0, \mathbf{r}_0$ are fixed in $\mathbb{R}^3$.

3. NOTATIONS AND DEFINITIONS

Let $\mathcal{O}_n^+$ denote the set of all rotations with determinant 1 in the n-dimensional Euclidean space $\mathbb{R}^n$, $\mathscr{E}_n^+$ denote the Euclidean group in n dimensions, and $\mathscr{G}$ the Galilei group. So,

$$\mathcal{O}_n^+ = \{\mathbf{R} | \mathbf{R}\mathbf{R}^T = \mathbf{1}, \det \mathbf{R} = 1\}$$

$$\mathscr{E}_n^+ = \{(\mathbf{R}, \mathbf{c}) | \mathrm{R} \in \mathcal{O}_n^+, \mathbf{c} \in \mathbb{R}^n\}$$

$$\mathscr{G} = \{(\mathbf{R}, \mathbf{v}, \mathbf{a}, t) | \mathbf{R} \in \mathcal{O}_n^+, \mathbf{v}, \mathbf{a} \in \mathbb{R}^3, t \in \mathbb{R}\}$$

The operations on $\mathscr{E}_n^+$ and $\mathscr{G}$ are, respectively,

$$\begin{aligned}(\mathbf{R}, \mathbf{c})(\bar{\mathbf{R}}, \bar{\mathbf{c}}) &= (\mathbf{R}\bar{\mathbf{R}}, \mathbf{R}\bar{\mathbf{c}} + \mathbf{c}) \\ (\mathbf{R}, \mathbf{v}, \mathbf{a}, \mathrm{t})(\bar{\mathbf{R}}, \bar{\mathbf{v}}, \bar{\mathbf{a}}, \bar{t}) &= (\mathbf{R}\bar{\mathbf{R}}, \mathbf{R}\bar{\mathbf{v}} + \mathbf{v}, \mathbf{R}\bar{\mathbf{a}} + \mathbf{a} + \mathbf{v}\bar{t}, t + \bar{t})\end{aligned} \tag{3.1}$$

The map $d: K \to K$ is a derivation on a field K (Lang, 1965, Chapter X.7) if

$$d(x+y) = d(x) + d(y) \tag{3.2a}$$

$$d(xy) = d(x)y + xd(y) \qquad \forall x, y \in K \tag{3.2b}$$

By using (3.2b), one deduces that $d(1) = 0$. For $K = \mathbb{R}$, one can conclude that

$$d(q) = 0 \qquad \forall q \in \mathbb{Q} \tag{3.3}$$

where $\mathbb{Q}$ is the set of all rational numbers. Proposition 10 in Chapter X of Lang, (1965) shows the existence of nonnull derivations.

Quaternion Representation of the Rotations

The quaternion representation of rotations is explained in many books. Let $\mathbf{R} := \mathbf{R}_p^{\mathbf{e}}$ be a three-dimensional rotation with axis $\mathbf{e}$ and angle p measured positively in the right-corkscrew direction. If $\mathbf{e} = e_1\mathbf{i} + e_2\mathbf{j} + e_3\mathbf{k}$ and $e \cdot e = 1$, the quaternion representation is

$$\mathbf{R}_p^{\mathbf{e}} \leftrightarrow [\boldsymbol{\xi}] = [\{\pm\boldsymbol{\xi}\} \tag{3.4}$$

where $\boldsymbol{\xi} = \cos\frac{1}{2}p + \sin\frac{1}{2}p(e_1\mathbf{i} + e_2\mathbf{j} + e_3\mathbf{k})$. If $\boldsymbol{\xi} = \xi_0 + \xi_1\mathbf{i} + \xi_2\mathbf{j} + \xi_3\mathbf{k}$ is a unit quaternion, then $\bar{\boldsymbol{\xi}} = \xi_0 - \xi_1\mathbf{i} - \xi_2\mathbf{j} - \xi_3\mathbf{k}$. We also have

$$\begin{gathered}
\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \qquad \mathbf{i}\mathbf{j} = \mathbf{k} = -\mathbf{j}\mathbf{i} \\
\mathbf{j}\mathbf{k} = \mathbf{i} = -\mathbf{k}\mathbf{j}, \qquad \mathbf{k}\mathbf{i} = \mathbf{j} = -\mathbf{i}\mathbf{k} \\
\boldsymbol{\xi}\bar{\boldsymbol{\xi}} = \bar{\boldsymbol{\xi}}\boldsymbol{\xi} = \xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 1 \\
(\overline{\boldsymbol{\xi}\boldsymbol{\eta}}) - \bar{\boldsymbol{\eta}}\bar{\boldsymbol{\xi}}, \qquad \mathbf{R}_p^{\mathbf{e}}\mathbf{v} = \boldsymbol{\xi}\mathbf{v}\bar{\boldsymbol{\xi}}
\end{gathered} \tag{3.5}$$

where $\boldsymbol{v} \in \mathbb{R}^3$.

Definition. Let d be a derivation of $\mathbb{R}$, and let $\boldsymbol{\eta} = \eta_0 + \eta_1\mathbf{i} + \eta_2\mathbf{j} + \eta_3\mathbf{k}$ be a quaternion which is not necessarily unit. Define

$$d(\boldsymbol{\eta}) := d(\eta_0) + d(\eta_1)\mathbf{i} + d(\eta_2)\mathbf{j} + d(\eta_3)\mathbf{k} \tag{3.6}$$

Then,

$$d(\bar{\boldsymbol{\eta}}) = \overline{d(\boldsymbol{\eta})} \tag{3.7a}$$

$$d(\boldsymbol{\eta} + \boldsymbol{\zeta}) = d(\boldsymbol{\eta}) + d(\boldsymbol{\zeta}) \tag{3.7b}$$

$$d(\boldsymbol{\eta}\boldsymbol{\zeta}) = d(\boldsymbol{\eta})\boldsymbol{\zeta} + \boldsymbol{\eta}d(\boldsymbol{\zeta}) \tag{3.7c}$$

$$\begin{aligned}
\boldsymbol{\eta}\mathrm{d}(\bar{\boldsymbol{\eta}}) = {} & \mathbf{i}(\eta_1 d(\eta_0) - \eta_0 d(\eta_1) + \eta_3 d(\eta_2) - \eta_2 d(\eta_3)) \\
& + \mathbf{j}(\eta_2 d(\eta_0) - \eta_0 d(\eta_2) + \eta_1 d(\eta_3) - \eta_3 d(\eta_1)) \\
& + \mathbf{k}(\eta_3 d(\eta_0) - \eta_0 d(\eta_3) + \eta_2 d(\eta_1) - \eta_1 d(\eta_2))
\end{aligned} \tag{3.7d}$$

after using (3.3).

4. AUTOMORPHISMS IN SPACE DIMENSION 3

Theorem 4. Let $\mathscr{E}_3^+$ be the Euclidean group in three dimensions with no space inversions, and let $\mathscr{G}$ be the corresponding Galilei group. Then $\mathscr{E}_3^+$ and $\mathscr{G}$ both have discontinuous automorphisms.

Proof. Using quaternion representation, we have

$$([\boldsymbol{\xi}], \mathbf{a})([\boldsymbol{\eta}], \mathbf{b}) = ([\boldsymbol{\xi}\boldsymbol{\eta}], \boldsymbol{\xi}\mathbf{b}\bar{\boldsymbol{\xi}} + \mathbf{a})$$

Let d be a nonnull derivation of $\mathbb{R}$. Define map A:

$$([\boldsymbol{\xi}], \mathbf{a}) \overset{A}{\mapsto} ([\boldsymbol{\xi}], \boldsymbol{\xi}d(\bar{\boldsymbol{\xi}}) + \mathbf{a}) \qquad \forall([\boldsymbol{\xi}], \mathbf{a}) \in \mathscr{E}_3^+ \tag{4.1}$$

Then A: $\mathscr{E}_3^+ \mapsto \mathscr{E}_3^+$ in view of (3.7d). It can be easily verified that A is an automorphism. Since d is nonnull, it can be checked that $\boldsymbol{\xi} d(\bar{\boldsymbol{\xi}})$ is not identically zero for all $\boldsymbol{\xi}$. The function $\boldsymbol{\xi} d(\bar{\boldsymbol{\xi}})$ is not identically zero for all ξ. The function $\boldsymbol{\xi} d(\bar{\boldsymbol{\xi}})$ is thus discontinuous, it being zero for $\boldsymbol{\xi}$ with rational components.

Similarly define map C:

$$([\boldsymbol{\xi}], \mathbf{v}, \mathbf{a}, \mathbf{t}) \overset{C}{\mapsto} ([\boldsymbol{\xi}], \mathbf{v}, \boldsymbol{\xi} d(\bar{\boldsymbol{\xi}}) + \mathbf{a}, t) \qquad \forall ([\boldsymbol{\xi}], \mathbf{v}, \mathbf{a}, t) \in \mathscr{G}$$

Then C is a discontinuous automorphism of $\mathscr{G}$ ■.

As stated in the introduction, the form of A in the proof of the last theorem is the form of a discontinuous map constructed by Tits (1970, Remark 11.3) for a different group.

Proof of Lemma 1. The statement of Lemma 1 is in Section 2 above. In this proof, we shall denote rotations by $\mathbf{R}$ as well as by its quaternion representation. This is for convenience. Details regarding this are in Section 3.

Let C: $\mathscr{E}_3^+ \mapsto \mathscr{E}_3^+$ be a discontinuous automorphism. From Lemmas 3.1 and 3.2 in Adeleke (1980), we deduce that there exists a continuous automorphism D such that

$$\begin{aligned} (1, \mathbf{z}) &\xmapsto{DC} (1, \mathbf{z}) \\ (\mathbf{R}_\alpha^{\mathbf{e}}, \mathbf{0}) &\xmapsto{DC} (\mathbf{R}_\alpha^{\mathbf{e}}, \mathbf{R}_\alpha^{\mathbf{e}} \mathbf{m}(\mathbf{e}, \alpha) - \mathbf{m}(\mathbf{e}, \alpha) + \phi(\mathbf{e}, \alpha)\mathbf{e}) \end{aligned} \tag{4.2}$$

where $\mathbf{m}(\mathbf{e}, \alpha) \cdot \mathbf{e} = 0$. *Composition of maps is read from right to left.* If we use the fact that $(\mathbf{R}_\pi^{\mathbf{e}}, 0)$ and $(\mathbf{R}_\alpha^{\mathbf{e}}, \mathbf{0})$ commute, we obtain $\mathbf{m}(\mathbf{e}, \alpha) = \mathbf{m}(\mathbf{e}, \pi)$. Let $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ be unit vectors in the x, y, and z directions, respectively. Commutativity of $\mathbf{R}_\pi^{\mathbf{i}}$, $R_\pi^{\mathbf{j}}$, and $\mathbf{R}_\pi^{\mathbf{k}}$ also implies that

$$\mathbf{m}(\mathbf{i}, \pi) = \mathrm{m}(\mathbf{j}, \pi) = \mathrm{m}(\mathbf{k}, \pi) = \mathbf{m}_0 \quad \text{(say)}$$

Therefore, if $\mathbf{e}$ is $\mathbf{i}$, $\mathbf{j}$, or $\mathbf{k}$, we have

$$\begin{gathered} (\mathbf{R}_\alpha^{\mathbf{e}}, \mathbf{0}) \xmapsto{EDC} (\mathbf{R}_\alpha^{\mathbf{e}}, \phi(\mathbf{e}, \alpha)\mathbf{e}) \\ (\mathbf{1}, \mathbf{z}) \xmapsto{EDC} (\mathbf{1}, \mathbf{z}) \end{gathered}$$

where E is an inner automorphism

$$E: \quad x \mapsto (\mathbf{1}, \mathbf{m}_0)x(\mathbf{1}, -\mathbf{m}_0), \qquad x \in \mathscr{E}_3^+$$

for all $x \in \mathscr{E}_3^+$. Define $A := EDC$. If $\bar{\mathbf{R}} = \mathbf{R}_\alpha^{\mathbf{g}}$, let $\mathbf{S}$ be a rotation for which $\mathbf{S}\mathbf{e} = g$. Then

$$\mathbf{S}\mathbf{R}_\alpha^{\mathbf{e}}\mathbf{S}^T = \mathbf{R}_\alpha^{\mathbf{g}} \tag{4.3}$$

If we use definition (4.2) and (4.3) to compute the image of $(\mathbf{R}_\alpha^{\mathbf{g}}, \mathbf{0})$, we arrive at $\phi(\mathbf{e}, \alpha) = \phi(\mathbf{g}, \alpha)$. Hence $\phi = \phi(\alpha)$. Now the image under A of the equation

$$(\mathbf{R}^{\mathbf{e}}_{\alpha_1+\alpha_2}, \mathbf{0}) = (\mathbf{R}^{\mathbf{e}}_{\alpha_1}, \mathbf{0})(\mathbf{R}^{\mathbf{e}}_{\alpha_2}, \mathbf{0})$$

gives

$$\phi(\alpha_1 + \alpha_2) = \phi(\alpha_1) + \phi(\alpha_2) \tag{4.4}$$

Furthermore, since $(\mathbf{R}^{\mathbf{e}}_{q\pi}, \mathbf{0})$ has finite order for every $q \in \mathbb{Q}$, its image satisfies

$$\phi(q\pi) = 0 \tag{4.5}$$

Define d: $[-1, 1] \to \mathbb{R}$ by

$$d(\cos \tfrac{1}{2}\alpha) = (\sin \tfrac{1}{2}\alpha)\phi(\alpha)$$

If $\cos \frac{1}{2}\alpha_1 = \cos \frac{1}{2}\alpha_2$, then $\frac{1}{2}\alpha_1 = 2n\pi \pm \frac{1}{2}\alpha_2$, and so $\sin \frac{1}{2}\alpha_1 = \pm \sin \frac{1}{2}\alpha_2$, and $\phi(\frac{1}{2}\alpha_1) = \pm\phi(\frac{1}{2}\alpha_2)$ after using (4.4) and $\phi(0) = 0$. It follows then that d is well defined. With this definition, we have

$$([\xi_0 + \mathbf{i}\xi_1], \mathbf{0}) \overset{A}{\mapsto} \left([\xi_0 + \mathbf{i}\xi_1], \frac{d(\xi_0)}{\xi_1}\mathbf{i}\right)$$

if $\xi_0 + \mathbf{i}\xi_1 \neq 1$.

I now show that d can be extended so that it becomes a derivation of $\mathbb{R}$. The image under A of the equation

$$([\xi_0 + \mathbf{i}\xi_1], \mathbf{0})([\xi_1 + \mathbf{i}\xi_0], \mathbf{0}) = ([0 + \mathbf{i}], \mathbf{0})$$

gives

$$\frac{d(\xi_0)}{\xi_1} = \frac{-d(\xi_1)}{\xi_0} \tag{4.6}$$

if $\xi_0 \neq 0 \neq \xi_1$. The image under A of the equation

$$([\xi_0 + \mathbf{i}\xi_1], \mathbf{0})([\eta_0 + \mathbf{j}\eta_1], \mathbf{0}) = ([\xi_0\eta_0 + \mathbf{i}\xi_1\eta_0 + \mathbf{j}\xi_0\eta_1 + \mathbf{k}\xi_1\eta_1], \mathbf{0})$$

gives

$$\begin{aligned} &\mathbf{i}\frac{d(\xi_0)}{\xi_1} + [\mathbf{j}(\xi_0^2 - \xi_1^2) + \mathbf{k}(2\xi_0\xi_1)]\frac{d(\eta_0)}{\eta_1} \\ &= \frac{d(\xi_0\eta_0)}{(1 - \xi_0^2\eta_0^2)^{1/2}} \frac{\mathbf{i}\xi_1\eta_0 + \mathbf{j}\xi_0\eta_1 + \mathbf{k}\xi_1\eta_1}{(1 - \xi_0^2\eta_0^2)^{1/2}} + \mathbf{c} \end{aligned}$$

where $\mathbf{c}$ is perpendicular to the other term on the right-hand side.

By taking components, one can deduce that

$$d(\xi_0\eta_0) = \eta_0 d(\xi_0) + \xi_0 d(\eta_0) \tag{4.7}$$

This result also holds if any of ξ_0 and η_0 is zero.

The image, under A, also of

$$([\xi_0+\mathbf{i}\xi_1],\mathbf{0})([\eta_0+\mathbf{i}\eta_1],\mathbf{0})=([\xi_0\eta_0-\xi_1\eta_1+\mathbf{i}(\xi_0\eta_1+\xi_1\eta_0)],\mathbf{0})$$

gives

$$\frac{d(\xi_0\eta_0-\xi_1\eta_1)}{\xi_0\eta_1+\xi_1\eta_0}=\frac{d(\xi_0)}{\xi_1}+\frac{d(\eta_0)}{\eta_1} \tag{4.8}$$

However, using (4.7), we obtain

$$\begin{aligned} d(\xi_0\eta_0)-d(\xi_1\eta_1)&=d(\xi_0)\eta_0+\xi_0\,d(\eta_0)-\xi_1\,d(\eta_1)-d(\xi_1)\eta_1\\ &=\frac{d(\xi_0)}{\xi_1}(\xi_0\eta_1+\xi_1\eta_0)+\frac{d(\eta_0)}{\eta_1}(\xi_0\eta_1+\xi_1\eta_0)\end{aligned}$$

after using (4.6). Statement (4.8) therefore implies

$$d(\xi_0\eta_0-\xi_1\eta_1)=d(\xi_0\eta_0)-d(\xi_1\eta_1)$$

and so

$$d(x_1+x_2)=d(x_1)+d(x_2) \tag{4.9}$$

for x_1,x_2 in a small neighborhood N of 0. The restriction to a small neighborhood of 0 is because of the restriction of the range of $(\xi_0\eta_0,\ \xi_1\eta_1)$.

If $|x_1|$, $|x_2|$, and $|x_1+x_2|$ are less than 1, there exists $r:=\cos p\pi$ for some $p\in\mathbb{Q}$ such that $rx_1, rx_2, r(x_1+x_2)\in N$. Since $d(rx_i)=rd(x_i)$, $i=1,2$, and $d(r(x_1+x_2))=rd(x_1+x_2)$ from (4.5) and (4.7), then (4.9) is valid for $x_1,x_2\in[-1,1]$ as long as $x_1+x_2\in[-1,1]$.

If $x_1,x_2\in[-1,1]$ and $x_1+x_2>1$, say, then x_1 and x_2 are positive and $x_1-1/2$, $x_2-1/2$, $x_1+x_2-1\in[-1,1]$. Then, from (4.9),

$$\begin{aligned} d(x_1+x_2-1)&=d(x_1-1/2)+d(x_2-1/2)\\ &=d(x_1)+d(x_2)\end{aligned}$$

using (4.5) and again (4.9). Similarly,

$$d(x_1+x_2+1)=d(x_1)+d(x_2)$$

if $x_1,x_2\in[-1,1]$ and $x_1+x_2<-1$. Now, extend $d:\mathbb{R}\to\mathbb{R}$ by

$$d(n+x)=d(x), \qquad x\in[-1,1], \quad n\in\mathbb{Z}$$

With the comments of the earlier part of this section, and results (4.9) and (4.7), it is routine to prove

$$\begin{aligned} &d(x_1+x_2)=d(x_1)+d(x_2)\\ &d(x_1x_2)=x_1d(x_2)+x_2d(x_1) \qquad \forall x_1,x_2\in\mathbb{R}\end{aligned} \tag{4.10}$$

Hence, d is a derivation of $\mathbb{R}$.

Since the automorphism A has the same effect on $(\mathbf{R}_\alpha^{\mathbf{e}}, \mathbf{0})$ for $\mathbf{e}=\mathbf{i}, \mathbf{j}$, or $\mathbf{k}$ and $\alpha \in \mathbb{R}$ as that of a typical automorphism of form (4.1), and since $\mathbf{R}_\alpha^{\mathbf{i}}, \mathbf{R}_\alpha^{\mathbf{j}}$, and $\mathbf{R}_\alpha^{\mathbf{k}}$ generate $\mathcal{O}_3^+$, then A is of the form (4.1). Define $B := (ED)^{-1}$. So, $C = BA$. ■

Proof of Lemma 2. As stated in the introduction, it was shown by Adeleke (1982) that discontinuous automorphisms exist for $\mathcal{G}$ if and only if they exist for $\mathscr{E}_3^+$. With Lemma 1, we can conclude from the body of proof of the Theorem in Adeleke (1982) that Lemma 2 holds. ■

Theorem 5. The cardinality of the set of all discontinuous automorphisms of $\mathscr{E}_3^+$ is $2^{2^{\aleph_0}}$.

Proof. Let T be a transcendental basis of $\mathbb{R}$ over $\mathbb{Q}$ (Lang, Chapter X, 1965). For every function $f: T \to \mathbb{R}$, there exists a unique derivation $d: \mathbb{R} \to \mathbb{R}$ such that $d(t_i) = f(t_i)$ for all $t_i \in T$.

I now indicate how this result follows from Theorem 7 and Proposition 10 of Chapter X of Lang (1965). For every $s \in \mathbb{R}$, let $d(s)$ be defined from the monic irreducible polynomial of s over $\mathbb{Q}(T)$ as indicated in Lang, (1965). Let an arbitrary pair $s_1, s_2 \in \mathbb{R}$ be given and let A be the set $\{s_1, s_2, s_1+s_2, s_1s_2\}$. Then the coefficients of the monic irreducible polynomial over $\mathbb{Q}(T)$ of each element of A are contained in $\mathbb{Q}(t_1, t_2, \ldots, t_r)$ for some finite $t_1, \ldots, t_r \in T$. Let $f(t_1), \ldots, f(t_r)$ be contained in an algebraic extension K of $\mathbb{Q}(t_1, t_2, \ldots, t_r)(t_{r+1}, \ldots, t_m)$ for some $t_{r+1}, \ldots, t_m$ distinct from $t_1, \ldots, t_r$. According to the said Proposition 10, there exists a derivation $\bar{d}$ of K such that $\bar{d}(t_i) = f(t_i)$, $i = 1, \ldots, r$, and $\bar{d}(t_j) = 0, j = r+1, \ldots, m$. The values of $\bar{d}$ for elements of A are also determined from their irreducible polynomials. By this construction, $\bar{d}$ and d are the same on A. Hence, equations (3.2) are satisfied by d for $x = s_1, y = s_2$. One then concludes that d is a derivation. That d is unique comes from T being a basis.

If $S = \{f \mid f: T \to \mathbb{R}\}$ and $\mathscr{D}$ is the set of all derivations of $\mathbb{R}$, then the cardinality $|S|$ of S is equal to the cardinality $|\mathscr{D}|$ of $\mathscr{D}$ by the above arguments. If T is countable, then $\mathbb{Q}(T)$ is countable. Since $\mathbb{R}$ is algebraic over $\mathbb{Q}(T)$, it would mean that $\mathbb{R}$ is countable, which is a contradiction. Hence, $|T| > \aleph_0$. Moreover, we then have $|T| = |\mathbb{Q}(T)| = |\mathbb{R}| = 2^{\aleph_0}$. Hence, using the notation of Halmos (1974), we have

$$|S| = |T^T| = 2^{2^{\aleph_0}}$$

By the Theorem in Adeleke (1980), every continuous automorphism of $\mathscr{E}_3^+$ is of the form

$$B: \quad (\mathbf{R}, \mathbf{c}) \to (\mathbf{R}_0\mathbf{R}\mathbf{R}_0^T, \alpha\mathbf{R}_0\mathbf{c} + \mathbf{R}_0\mathbf{R}\mathbf{R}_0^T\mathbf{b}_0 - \mathbf{b}_0)$$

where $\mathbf{R}_0$ is some rotation, $\mathbf{b}_0$ is a vector in $\mathbb{R}^3$, and α is a nonzero number.

This, with Lemma 1 and the result on $|S|$ above, concludes the proof of this lemma. ■

5. AUTOMORPHISMS IN HIGHER DIMENSIONS

I first introduce some notations and definitions. Let

$$K:=\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$$

be a given fixed orthonormal basis in $\mathbb{R}^n$. Let i, j, and k be distinct indices in $\{1, 2, \ldots, n)\}$. The symbol $\mathbf{R}_{ij}$ stands for the $(n-2, 2)$ involution for which

$$\mathbf{R}_{ij}\mathbf{e}_l = \begin{cases} \mathbf{e}_l & \text{if} \quad i \neq l \neq j \\ -\mathbf{e}_l & \text{if} \quad l = i \text{ or } j \end{cases}$$

An r-rotation $(r \leq n)$ is a rotation in $\mathcal{O}_n^+$ which fixes all $\mathbf{e}_i$ except possibly r of them. Let $\zeta_{ijk}:=\{S_{ijk}\}$ denote the set of rotations of order 2 which fix all $\mathbf{e}_m$ except possibly $\mathbf{e}_i$, e_j, $\mathbf{e}_k$.

The following lemma is essentially an extension of Lemma 3.1 in Adeleke (1980) for $n=3$.

Lemma 3. Let $n \geq 4$. Given an automorphism $D: \mathscr{E}_n^+ \to \mathscr{E}_n^+$, there exists a continuous automorphism B of $\mathscr{E}_n^+$ such that

$$\begin{aligned} (\mathbf{1}, \mathbf{d}) &\xmapsto{BD} (\mathbf{1}, \mathbf{d}) \\ (\mathbf{R}_{ij}, \mathbf{0}) &\xmapsto{BD} (\mathbf{R}_{ij}, \mathbf{0}) \\ (\mathbf{R}, \mathbf{0} &\xmapsto{BD} (\mathbf{R}, \mathbf{c}(\mathbf{R})), \qquad \mathbf{R} \in \mathcal{O}_n^+ \end{aligned} \tag{5.1}$$

where $\mathbf{c}(\mathbf{R})$ is a function of $\mathbf{R}$.

Proof. Define

$$U:=\{(\mathbf{R}, \mathbf{d}) \in \mathscr{E}_n^+ | \mathscr{C}((\mathbf{R}, \mathbf{d})) \subseteq C((\mathbf{R}, \mathbf{d}))\}$$

$$\mathscr{C}(x):=\{y^{-1}xy | y \in \mathscr{E}_n^+\}$$

$$C(x):=\{y \in \mathscr{E}_n^+ | yx = xy\}$$

i.e., $\mathscr{C}(x)$ and $C(x)$ denote the conjugacy class and centralizer subgroup of x in $\mathscr{E}_n^+$, respectively. Then $U=\{(\mathbf{1}, \mathbf{b}) | \mathbf{b} \in \mathbb{R}^n\}$, and is thus an invariant subgroup. The method of proof in the case $n=3$ in Lemma 3.1 of Adeleke (1980) then applies exactly to this case and yields

$$(\mathbf{1}, \mathbf{b}) \overset{D}{\mapsto} (\mathbf{1}, \alpha \mathbf{R}_0 \mathbf{b})$$

$$(\mathbf{R}, \mathbf{0}) \overset{D}{\mapsto} (\mathbf{R}_0 \mathbf{R} \mathbf{R}_0^T, \bar{\mathbf{c}}(\mathbf{R}))$$

for some $\mathbf{R}_0 \in \mathcal{O}_n^+$ and $\alpha \in \mathbb{R}$, $\alpha \neq 0$. So there exists a continuous automorphism E such that

$$(\mathbf{1}, \mathbf{d}) \xmapsto{ED} (\mathbf{1}, \mathbf{d})$$

$$(\mathbf{R}, \mathbf{0}) \xmapsto{ED} (\mathbf{R}, \hat{\mathbf{c}}(\mathbf{R}))$$

Note that $\{\mathbf{R}_{12}, \mathbf{R}_{23}, \ldots, \mathbf{R}_{n-1,n}\}$ generates $\{\mathbf{R}_{ij}\}$. Suppose (ED)-$((\mathbf{R}_{ij}, 0)) = (\mathbf{R}_{ij}, \mathbf{b}_{ij})$. Commutativity of $\mathbf{R}_{ij}, \mathbf{R}_{jl}$ shows that we can write $\mathbf{b}_{ij} = \mathbf{b}_i + \mathbf{d}_j$ for some $\mathbf{b}_1, \ldots, \mathbf{b}_n$. Let F be the continuous automorphism defined by

$$\mathbf{x} \in \mathscr{E}_n^+ \overset{F}{\mapsto} (\mathbf{1}, -\mathbf{z}_0)\mathbf{x}(\mathbf{1}, \mathbf{z}_0)$$
$$\mathbf{z}_0 = \tfrac{1}{2}(\mathbf{b}_1 + \mathbf{b}_2 + \cdots + \mathbf{b}_n)$$

Then

$$(\mathbf{1}, \mathbf{d}) \xmapsto{FED} (\mathbf{1}, \mathbf{d})$$

$$(\mathbf{R}, \mathbf{0}) \xmapsto{FED} (\mathbf{R}, \mathbf{c}(\mathbf{R}))$$

$$(\mathbf{R}_{ij}, \mathbf{0}) \xmapsto{FED} (\mathbf{R}_{ij}, \mathbf{0})$$

for some function $\mathbf{c}(\mathbf{R})$. Put $B := FE$. ■

Theorem 6. If $n \geq 4$, all automorphisms of $\mathscr{E}_n^+$ are continuous.

Proof. Let D be an automorphism of $\mathscr{E}_n^+$ and let B be the continuous automorphism satisfying (5.1). If $\mathbf{S}_{123} \in \zeta_{123}$, and $(BD)((\mathbf{S}_{123}, \mathbf{0})) = (\mathbf{S}_{123}, \mathbf{z})$, then $\mathbf{S}_{123}\mathbf{z} = -\mathbf{z}$, since $\mathbf{S}_{123}$ is of order 2. $S_0, \mathbf{z}$ is a linear combination of $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ from the definition of ζ_{123}. Consequently, $\mathbf{R}_{12}\mathbf{R}_{34}\mathbf{z} = -\mathbf{z}$.

On the other hand, the product $\mathbf{R}_{12}\mathbf{R}_{34}$ commutes with $\mathbf{S}_{123}$. Hence, their images $(\mathbf{R}_{12}\mathbf{R}_{34}, \mathbf{0})$ and $(\mathbf{S}_{123}, \mathbf{z})$ commute. Thus, $\mathbf{R}_{12}\mathbf{R}_{34}\mathbf{z} = \mathbf{z}$. This, with the last equation in the previous paragraph above, shows $\mathbf{z} = \mathbf{0}$; and so,

$$(\mathbf{S}_{123}, \mathbf{0}) \xmapsto{BD} (\mathbf{S}_{123}, \mathbf{0})$$

By similar reasoning, we infer that BD fixes any $(\mathbf{S}_{ijk}, \mathbf{0})$.

From well-known results on rotations [see, e.g., Adeleke (1980), equation (3.14)], we deduce that $BD((\mathbf{R}, \mathbf{0})) = (\mathbf{R}, \mathbf{0})$ for every 3-rotation. Since the set of all k-rotations generates the $(k+1)$-rotations, we see by induction that $BD((\mathbf{R}, \mathbf{0})) = (\mathbf{R}, \mathbf{0})$ for all $\mathbf{R} \in \mathcal{O}_n^+$. Hence BD is the identity. ■

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